

# On the Brauer monoid for finite fields

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## Abstract

The definition of the Brauer monoid was given in [3]. In this article it is studied by the notions of modifications [6] and 0-cohomology [5]. We investigate the impact of invertible elements of modifications on the structure of the Brauer monoid, especially for finite fields.

## 1 Introduction

It is well-known that the Brauer group of any finite field is trivial [2]. Therefore the so called Brauer monoid proposed in [3] is of interest. This monoid generalizes the Brauer group and isn't trivial for any non-trivial field extension. One can hope that the studying of its properties will be useful for the investigation of algebras over finite fields.

The description of the Brauer monoid by modifications and their 0-cohomology proposed in [6] is more convenient in our opinion than original one [3]. We give in Section 2 this description and a necessary information about semigroup 0-cohomology too.

Section 3 is devoted to the proof of a theorem, which facilitates essentially the calculation of a Brauer monoid for finite fields by elimination of the invertible elements.

Finally we note that at stretch of this article relative Brauer monoids (and relative Brauer groups) are considered only, so the adjective "relative" will be omitted.

## 2 Preliminary: 0-cohomology and modifications

Semigroup 0-cohomology is a specific case of partial cohomologies which were built in [5]; it had appeared in the investigation of the projective representations of semigroups.

Let  $S$  be an arbitrary semigroup with a zero. An Abelian group  $A$  is called a *0-module* over  $S$ , if an action  $(S \setminus 0) \times A \rightarrow A$  is defined which satisfies for all  $s, t \in S \setminus 0$ ,  $a, b \in A$  the following conditions:

$$\begin{aligned} s(a + b) &= sa + sb, \\ st \neq 0 &\Rightarrow s(ta) = (st)a. \end{aligned}$$

A *n-dimensional 0-cochain* is a partial  $n$ -place mapping out of  $S$  to  $A$  which is defined for all  $n$ -tuples  $(s_1, \dots, s_n)$ , such that  $s_1 \cdot \dots \cdot s_n \neq 0$ . The coboundary operator is given like for the usual cohomology by the formula

$$\begin{aligned} \partial^n f(s_1, \dots, s_{n+1}) &= s_1 f(s_2, \dots, s_{n+1}) + \sum_{i=1}^n (-1)^i f(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1}) \\ &+ (-1)^{n+1} f(s_1, \dots, s_n) \end{aligned}$$

The equality  $\partial^2 = 0$  is valid too; obtained cocycles (cohomology) are called *0-cocycles* (*0-cohomology*) and their groups are denoted by  $Z_0^n(S, A)$  (resp.  $H_0^n(S, A)$ ).

Note that for a semigroup  $T^0 = T \cup 0$  with an adjoined zero

$$H_0^n(T^0, A) \cong H^n(T, A),$$

so 0-cohomology may be considered as a generalization of Eilenberg – MacLane cohomology.

Let  $L$  be a finite-dimensional normal extension of a field  $K$  with the Galois group  $G$ ,  $L^\times$  be the multiplicative group of  $L$ .

By a *modification*  $G(\star)$  of the group  $G$  we mean a semigroup on the set  $G^0 = G \cup 0$  with operation  $\star$  such that  $x \star y$  is equal either to  $xy$  or to 0, while

$$0 \star x = x \star 0 = 0 \star 0 = 0$$

and the identity of  $G$  is the same for the semigroup  $G(\star)$ .

In other words, to obtain a modification, one must erase the contents of some inputs in the multiplication table of  $G$  and insert there zeros so that the new operation would be associative.

Note some general properties of modifications. Firstly, a modification of  $G$  satisfies the weak cancellation condition: from  $x \star z = y \star z \neq 0$  it follows  $x = y$  and analogously for left cancellation. Secondly, let  $U$  be the subgroup of invertible elements in  $G(\star)$ . Then its complement  $I = G(\star) \setminus U$  is a two-sided ideal. It follows from finiteness of  $G$  that  $I$  is nilpotent[6].

$L^\times$  is a 0-module over every modification, where elements of the modification act on  $L^\times$  as automorphisms of the field. 0-cohomology groups  $H_0^2(G(\star), L^\times)$  will be called *components of the Brauer monoid*. In the case when operation  $\star$  is defined by such a way that  $x \star y = xy$  for  $x, y \neq 0$ , the component of the Brauer monoid turns out the Brauer group.

Let  $S = G(\star)$  and  $T = G(*)$  be modifications of  $G$ . We write  $S \prec T$ , if  $x \star y = 0$  implies  $x \star y = 0$  for all  $x, y \in G$ . Clearly, in this case a homomorphism is defined

$$\varepsilon_{T,S} : H_0^2(T, L^\times) \longrightarrow H_0^2(S, L^\times)$$

Since for  $S \prec T \prec U$  these homomorphisms yield the equalities

$$\varepsilon_{U,T} \varepsilon_{T,S} = \varepsilon_{U,S}, \quad \varepsilon_{S,S} = id,$$

one can build by the standard way [1] the semilattice of groups  $H_0^2(S, L^\times)$  (where  $S$  runs over all modifications of  $G$ ), which is called a *(relative) Brauer monoid*  $M(G, L)$ .

### 3 Invertible elements in modifications

As above let  $S = G(\star)$  be a modification of the Galois group  $G$ ,  $U$  the subgroup of invertible elements of  $S$ ,  $I = S \setminus U$ . We shall write in this section  $xy$  instead of  $x \star y$ , inasmuch as the operation of the group  $G$  will not be used. Besides let us agree to employ additive notation for the  $G$ -module  $L^\times$ .

The inclusion  $U \hookrightarrow G$  induces a homomorphism

$$\varphi : H_0^2(S, L^\times) \rightarrow H^2(U, L^\times)$$

We study this homomorphism in the situation when  $U$  is a normal subgroup of  $G$ . Then  $U$  turns out normal in  $S$  too (in the meaning that  $xU = Ux$  for all  $x \in S$ ). The partition into cosets of  $U$  (together with  $\{0\}$ ) is a congruence on  $S$ . The quotient-semigroup on this congruence will be denoted by  $S/U$ .

Further, if  $U \triangleleft S$  then the multiplicative group of the subfield  $P$  of all  $U$ -fixed elements is a  $S/U$ -module. The inclusion  $P^\times \hookrightarrow L^\times$  and the epimorphism  $S \rightarrow S/U$  induce a homomorphism

$$\psi : H_0^2(S/U, P^\times) \rightarrow H_0^2(S, L^\times)$$

**Theorem 3.1** *Let  $U \triangleleft S$ . Then the sequence*

$$0 \longrightarrow H_0^2(S/U, P^\times) \xrightarrow{\psi} H_0^2(S, L^\times) \xrightarrow{\varphi} H^2(U, L^\times)$$

*is exact.*

**Proof.** We fix a system  $T$  of representatives of the cosets with respect to  $U$ :  $S = \{0\} \cup (\bigcup_{t \in T} Ut)$  (as usually the representative for  $U$  is the identity). In what follows we denote by  $a, b, c$  elements of  $U$ , by  $x, y, z$  elements of  $I \setminus 0$ , by  $r, s, t$  elements of  $T$ . If  $f$  is a (0)-cocycle then the respective element of the (0)-cohomology group is denoted by  $[f]$ .

A remark must yet be made. It follows out of normality of  $U$  that for every  $a \in U$ ,  $x \in I \setminus 0$  there is such  $a_x \in U$  that  $ax = xa_x$ . In addition  $a_x$  is defined uniquely because  $xa_x \neq 0$ . Next, if  $a$  runs over  $U$  then  $a_x$  runs over it too, and  $(ab)_x = a_x b_x$  for  $a, b \in U$ .

Let  $[f] \in \text{Ker } \varphi$  where  $f$  is a 0-cocycle of  $Z_0^2(S, L^\times)$ . One can assume that  $f(a, b) = 0$  for all  $a, b \in U$ . Consider the cochain of the group  $U$

$$\pi_x(a) = f(a, x) - f(x, a_x).$$

It is a cocycle because

$$\begin{aligned} \partial \pi_x(a, b) &= af(b, x) - af(x, b_x) - f(ab, x) + f(x, a_x b_x) + f(a, x) - f(x, a_x) \\ &= \partial f(a, b, x) - \partial f(a, x, b_x) - f(ax, b_x) + f(x, a_x b_x) - f(x, a_x) \\ &= -f(xa_x, b_x) + f(x, a_x b_x) - f(x, a_x) = \partial f(x, a_x, b_x) = 0 \end{aligned}$$

Since  $H^1(U, L^\times) = 0$  (see, e. g., [4]), it follows from this that

$$\pi_x(a) = (a - 1)\lambda(x),$$

where  $\lambda(x) \in L^\times$  for all  $x \in I \setminus 0$ . Set  $\lambda(a) = 0$  and  $g = f + \partial \lambda$ . Then  $g(a, b) = 0$  and besides

$$\begin{aligned} g(a, x) &= f(a, x) - a\lambda(x) + \lambda(ax) \\ &= \pi_x(a) + f(x, a - x) - a\lambda(x) + \lambda(xa_x) \\ &= f(x, a - x) - \lambda(x) + \lambda(xa_x) = g(x, a_x). \end{aligned}$$

Next let us set  $\rho(at) = g(a, t)$  for  $t \in T$  and consider the 0-cocycle  $h = g + \partial\rho$ . Then

$$h(a, bt) = g(a, bt) + ag(b, t) - g(ab, t) = \partial g(a, b, t) + g(a, b) = 0 \quad (1)$$

$$h(at, b) = g(at, b) - g(t, a_x b) + g(a, t) = \partial g(t, a_x, b) - tg(a_x, b) = 0 \quad (2)$$

From here we obtain for  $xy \neq 0$ :

$$\begin{aligned} ah(x, y) &= h(ax, y) - h(a, xy) + h(a, x) = h(xa_x, y) \\ &= xh(a_x, y) + h(x, a_x y) - h(x, a_x) = h(x, a_x y) = h(x, y(a_x)_y) \\ &= -xh(y, (a_x)_y) + h(xy, (a_x)_y) + h(x, y) = h(x, y) \end{aligned}$$

Hence  $h(x, y) \in P^\times$ . Besides the last calculation implies

$$h(ax, y) = h(x, a_x y) = h(x, y)$$

Since  $a$  and  $a_x$  both run over all group  $U$  we have:

$$h(ax, y) = h(x, ay) = h(x, y)$$

This means that  $h$  defines a 0-cocycle  $\bar{h} \in Z_0^2(S/U, P^\times)$  by the next way:

$$\bar{h}(Us, Ut) = h(s, t) \text{ for } st \neq 0.$$

So for a given 0-cocycle  $f \in Z_0^2(S, L^\times)$  by the 0-cocycles  $g$  and  $h$  which are cohomological to it, we construct the 0-cocycle  $\bar{h} \in Z_0^2(S/U, P^\times)$ .

We show that the correspondence  $h \rightarrow \bar{h}$  extends to the cohomology mapping. Let  $h = \partial\sigma$  for some 0-cochain  $\sigma \in C_0^1(S, L^\times)$ . Since  $\partial\sigma(a, b) = h(a, b) = 0$ , one has  $\sigma(a) = (a - 1)\mu$  for the restriction of  $\sigma$  on  $U$ , where  $\mu \in L$ . Further, it follows out of (1) and (2)  $\partial\sigma(a, x) = \partial\sigma(x, a) = 0$ . Therefore

$$\begin{aligned} a[\sigma(x) - (x - 1)\mu] &= \partial\sigma(a, x) + \sigma(ax) - \sigma(a) - xa_x\mu + a\mu \\ &= \sigma(xa_x) - xa_x\mu + \mu \\ &= \partial\sigma(x, a_x) - x\sigma(a_x) + \sigma(x) - xa_x\mu + \mu \\ &= \sigma(x) - (x - 1)\mu, \end{aligned}$$

so  $\sigma(x) - (x - 1)\mu \in P^\times$ . Let  $\tau(g) = \sigma(g) - (g - 1)\mu$  for any  $g \in S \setminus 0$ . Then  $\partial\tau = \partial\sigma = h$ ,  $\tau(g) \in P^\times$  and in addition

$$\begin{aligned} \tau(ax) &= -\partial\sigma(a, x) + a\sigma(x) + \sigma(a) - (ax - 1)\mu \\ &= a[\tau(x) - (x - 1)\mu] + (a - 1)\mu - (ax - 1)\mu \\ &= a\tau(x) = \tau(x), \end{aligned}$$

i. e. 0-cochain  $\tau$  is constant on cosets of  $U$ . Setting  $\bar{\tau}(Ut) = \tau(t)$  we get  $\bar{h} = \partial\bar{\tau}$ . Thus a homomorphism  $\text{Ker}\varphi \rightarrow H_0^2(S/U, P^\times)$  is defined.

Now we construct an inverse map. Let  $\bar{h} \in Z_0^2(S/U, P^\times)$ . Set  $h(at, b) = h(a, bt) = 0$  and  $h(as, bt) = \bar{h}(Us, Ut)$  for  $st \neq 0$ . Then one can verify straightforward that  $\partial h = 0$ , and since the restriction of  $h$  on  $U$  equals zero,  $[h] \in \text{Ker}\varphi$ .

Let  $\bar{h} = \partial\bar{\gamma}$  for some 0-cochain  $\bar{\gamma} \in C_0^1(S/U, P^\times)$ . Setting  $\gamma(at) = \bar{\gamma}(Ut)$  we get  $h = \partial\gamma$ . Thus we constructed the sought mapping  $\text{Ker}\varphi \rightarrow H_0^2(S/U, P^\times)$  and proved that these groups are isomorphic. ■

**Remark 1.** The proved theorem generalizes the results from [3] (where one assumed that  $I^2 = 0$ ) and [6] (where one assumed that the modification  $S$  is commutative).

**Remark 2.** Indeed  $S/U$  is a modification of the group  $G/U$ , the Galois group of the extension  $P/K$ , so  $H_0^2(S/U, P^\times)$  is a component of the respective Brauer monoid  $M(G/U, P)$ .

**Corollary 3.1** *If the field  $L$  is finite then*

$$H_0^2(S, L^\times) \cong H_0^2(S/U, P^\times)$$

**Proof.** Since in this case the group  $G$  is Abelian (even cyclic) so  $U \triangleleft S$ . Then  $U$  is the Galois group of  $L/P$  and  $H^2(U, L^\times)$  is trivial as a Brauer group of a finite field. ■

## References

- [1] A. H. Clifford, G. B. Preston. *Algebraic Theory of Semigroups*, Amer. Math. Soc., Providence, 1964.
- [2] Ju. A. Drozd, V. V. Kirichenko. *Finite Dimensional Algebras*. Springer-Verlag, Berlin-Heidelberg-New-York, 1994.
- [3] D. E. Haile, R. G. Larson, M. E. Sweedler. *A new invariant for  $\mathbf{C}$  over  $\mathbf{R}$* . Amer. J. Math., **105**(1983), N3, p.689-814.
- [4] S. Lang. *Algebra*, Addison-Wesley Publ. Comp., 1965.
- [5] B. V. Novikov. *On partial cohomologies of semigroups*. Semigroup Forum, **28**(1984), N1-3, pp.355-364.
- [6] B. V. Novikov. *The Brauer monoid*. Matem. zametki, **57**(1995), No. 4, pp.633-636 (Russian). Translated in: Math. Notes **57**(1995), No. 3-4, pp.440-442.

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